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On the theory of vortex quantum tunnelling in the dense Bose superfluid helium II

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Abstract

The quantum tunnelling and nucleation theory of vortices in helium II is reviewed. Arguments are given that the only reliable method to calculate tunnelling probabilities in this highly correlated, strongly interacting many-body system is the semiclassical, large scale approach for evaluation of the tunnelling exponent, which does not make any assumptions about the unknown dynamical behaviour of the fluid on microscopic scales. The geometric implications of this semiclassical theory are represented in some detail and its relevance for the interpretation of experimental data is discussed.

Key words: Quantum tunnelling, Vortices, Dense Superfluid

1 Preface

The nucleation theory of quantized vortices in the Bose superfluid helium II has been an elusive subject ever since the existence of quantized vortices was conjectured by Lars Onsager in 1949. The difficulty to grasp their coming into existence in a definite manner from first principles has one fundamental reason: There is no microscopic theory of this superfluid. We do not know how to describe the motion of a vortex on scales of the order of the coherence length, where this motion is governed by the full quantum many-body structure of the superfluid and the interaction of the vortex with the microscopic excitations. Consequently, we can not follow the vortex on any stage of the evolution from a virtual vortex to a mature one describable by semiclassical means. At present,

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even the many-body structure and energy of a simple static rectilinear vortex can not be uniquely determined, let alone that of a circular vortex which is still beyond reach even for Monte Carlo calculations [1].

This lack of a microscopic idea of vortex motion makes it necessary to resort to a semiclassical, hydrodynamic (large scale) theory. In such a treatment, the existence of the vortex as a semiclassical object has to be assumed *ab initio*. No details of the underlying microscopic dynamics, *i.e.* of the actual nucleation event, are to be described in such a theory, but only the laws which rule vortex motion on curvature scales well beyond the core size ξ (Fig. 1). This is admissible because in helium II this length scale is of the order of the interatomic spacing and the relevant scales for potential barriers to be surmounted by thermal activation or crossed by quantum mechanical tunnelling usually have spatial extent and length scales far beyond ξ. Other advantages of ^{4}He include that it can be prepared to have very high purity such that dissipation has negligible influence. The order parameter ϕ has the simplest conceivable structure resulting from a spontaneously broken U(1) symmetry. In contrast to the extraordinarily complicated behaviour on ξ-scales this superfluid can be studied under quite basic and simple assumptions if one is remaining completely in the semiclassical, large scale domain.

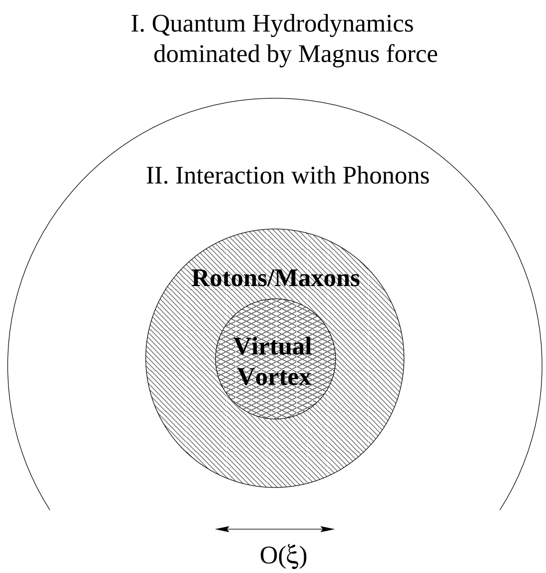

Fig. 1. Length scales of vortex motion and interaction with elementary excitations in the dense superfluid helium II. It is only in the regions I and II where a semiclassical picture of vortex motion is applicable. The vortex in the central region of extension O(ξ) is termed virtual because it is no well-defined topological object in this region.

In this paper, we will discuss and investigate more closely the hydrodynamic

theory of vortex quantum tunnelling. In the section which follows we contrast it with different other ways to calculate tunnelling rates. Then we explain the way in which the classical theory of the tunnelling exponent in three dimensions is a geometric theory in section 3. How to obtain an explicit solution of the boundary problem for a point vortex moving around an ellipse in two dimensions is demonstrated in section 4. It is shown that at boundaries with strongly varying curvature radii strict bounds on semiclassicality of vortex motion are required, which lead in turn to restrictions on the tunnelling energy of the vortex. We estimate the prefactor of the tunnelling rate in section 5 and the relation to available experimental data is established. We conclude with some general remarks. The discussion will be restricted to zero temperature as we will be interested in pure quantum mechanical tunnelling without any assistance of thermal fluctuations.

2 Different approaches

We present in this section a critical overview of some of the methods developed over the last decades to describe vortex quantum nucleation and tunnelling.

The description of the tunnelling phenomenon by means of classical incompressible hydrodynamics for the vortex motion was first undertaken in [2]. The motion of a vortex half-ring in the presence of a half-sphere at an otherwise flat boundary was considered, which is solvable analytically because of maximal symmetry by using a Legendre function of the second kind. The method is exact insofar as the (pinning) potential in which the vortex moves is exactly known in its relation to known geometrical quantities. This procedure involves no assumptions about the dynamics of the order parameter. The only fundamental ingredient is the existence of the Magnus force acting on the vortex. This is essentially the approach which we will investigate in this paper.

The approach of [3] consisted in evaluating transition probabilities between an initial many-particle wave function without and a final one with vortex. The many-particle wave functions were constructed from single particle solutions of the Gross-Pitaevskiĭ equation [4], valid for a weakly interacting, dilute Bose gas. The main difference to [2], as discussed in [5], consisted in the existence of an (intermediate) deformed vortex ring state with a depression of the flow velocity out of the ring plane, leading to a smaller vortex energy. This also led to a different form of the tunnelling exponent by a logarithmic factor. Ultimately, though, this theory gave approximately the same value for the exponent as the collective co-ordinate approach of [2], provided one takes values of tunnelling parameters typically realized.

In the recent past, there has been put some effort into numerical calculations

of instantons [6,7]. The instanton is a finite action solution to the Euclidean equations of motion and describes imaginary time motion under the tunnelling barrier. In [7], an effective Lagrangian for a massive elastic string moving in a pinning potential near criticality was derived to calculate the shape of the instanton. The authors of [6], making use of the Gross-Pitaevskiĭ equation, examined the nucleation of vortices in the streaming motion past an obstacle by solving the full nonlinear field equation in 1+1d. The intent there was to investigate the dynamics of the vortex linked to that of the condensate. With regard to this approach, it should be emphasised that use of the Gross-Pitaevskiĭ equation for helium II on small (ξ) scales can at best claim phenomenological correctness, in this way comparable to density-phase functional theories [8].

Field-theoretical calculations, akin to those of pair creation in quantum electrodynamics, but for the corresponding one-dimensionally extended, closed object vortex ring in a U(1) superfluid, were performed in [9–12]. The antisymmetric gauge field tensor in the dual formulation of string dynamics [13] has been employed and the components of the 3-form field strength tensor identified with stringy generalizations of magnetic and electric fields. The theory is 'relativistic' in the sense that the propagation velocity of light is replaced by that of sound in Lorentz invariant wave equations. The main problem of this formulation can be ascribed to the usage of an effective Nambu-Goldstone Lagrangian for sound on scales of O(ξ). This Lagrangian does not describe the real superfluid on these scales. In particular, because vortex velocities approach the speed of sound c_s, these approaches give the *hydrodynamic*, *i.e.* large scale, mass of the vortex undue weight in the tunnelling exponent [14,15].

In summary, all these approaches have in common that they are *mean-field* theories, assuming that the quantum mechanical fluctuations of the U(1) field $\hat{\phi}(\vec{x},t)$ under consideration are on any scale relatively small compared to its expectation value $\phi =< \hat{\phi}(\vec{x},t) >$, the order parameter. Such theories can not be derived in ^{4}He from first principles on arbitrary length scales, because this dense superfluid is a strongly coupled system, in which quantum fluctuations play a major role. On the atomic scale of the coherence length this implies, for example, that the quantum mechanical uncertainty for the density operator is of the same order as the local value of the density itself, the expectation value of this operator. Put very simple: One does not know (for purely quantum statistical reasons) if the particle(s) in a coherence length sized volume is (are) there or not. The variable conjugate to density, the phase, has no definite value in such a small volume as well. In other words still, a collective central vortex co-ordinate of the quantum field ceases to be meaningful for the description of vortex motion on these small scales.

3 Semiclassical vortex quantum tunnelling as a geometric theory

The aforementioned reservations about various (mainly field-theoretical) approaches led the present author to a re-investigation [17] of the method [2] put forward already in 1972. The most appealing feature of this method is its simplicity and exactness in the semiclassical limit (cf. regions I and II in Fig. 1). There is no reason for the description of the vortex as a stringlike entity in terms of a collective co-ordinate not to be a viable candidate for the investigation of vortex quantum tunnelling, if the requirements for this formulation are carefully met. It should have become apparent from the above discussion that this method is in fact at present the only fully reliable one for the calculation of observable tunnelling probabilities in the actual, dense superfluid helium II.

3.1 The vortex action for a string

To set up the general framework for the description of a vortex string in the hydrodynamic limit, it is convenient to use the dual formulation of string dynamics in terms of the antisymmetric tensor gauge field $b_{\mu\nu}$ already mentioned above in the field-theoretical context [16]. The appendix A contains a short summary of this formulation. The simplest way to get an intuitive picture of the dual formulation is that the tensor gauge field is a relativistic generalization of the stream function of classical hydrodynamics [18], *i.e.* it is chosen such that its (generalized) rotation is the velocity field of the fluid. To begin with and introduce further into this formalism as far as we need it in what follows, we then recall in addition that a semiclassical point vortex is analogous to a charged particle in a magnetic field. The stream function ψ of classical hydrodynamics plays the part of the negative scalar potential $-a^0 \equiv -\Phi$ acting on the vortex with a charge $q = (m\rho_0)^{1/2}\kappa$. The vector potential a_i stems from the rotation of a 'magnetic' field which is antiparallel to the local circulation vector and related to the superfluid density. These *analoga* can be readily generalized to a stringlike entity by defining the above quantities locally on the string [16]. The Magnus force, a stringy generalization of the Lorentz force, is then derived in a Galilei invariant superfluid from the variation of the action

$$\begin{aligned} S_M &= m\rho_0\kappa \int\!\!\oint dt\, d\sigma \left(\psi_i X'^i + b_{ij} X'^j \dot{X}^i\right) \\ &\equiv q \int\!\!\oint dt\, d\sigma \left(-\Phi + a_i \dot{X}^i\right) \end{aligned} \tag{1}$$

with respect to the vortex co-ordinate $X^i(t,\sigma)$ (we employ the convention of summation over equal indices). The line tangent is denoted $X'^i \equiv \partial X^i/\partial\sigma$ and the vortex velocity $\dot{X}^i \equiv \partial X^i/\partial t$. The bulk number density and the quantum

of circulation bear the symbols ρ_0 and κ, respectively. The parameter σ labels points on the string.

As expressed in the second line of equation (1), the first part of the action S_M is the generalization of the negative scalar potential in usual (non-relativistic) electrodynamic actions. The second part is analogous to the linear coupling term of the gauge vector potential $a_i = (m\rho_0)^{1/2} b_{ij} X'^j$ to the vortex velocity $\dot{X}^i$. The generalization from the point particle to the string essentially consists in introducing locality in σ and admitting that the line tangent can point in any direction. The stream function becomes a vector with components $\psi_i = b_{0i}$ along the tangent vector. An isotropic gauge choice for the purely spatial part of $b_{\mu\nu}$ in Cartesian co-ordinates is given by $b_{ij} = -(1/d) n_{ijk} X^k$, where d signifies the spatial dimension and n_{ijk} the unit antisymmetric symbol. This represents the isotropic solution of equation (A.4).

The local canonical momentum of a vortex per σ-length in a *compressible* superfluid

$$P_i = P_i^{\text{inc}} + P_i^{\text{kin}} = m\rho_0\kappa\, b_{ij} X'^j + \sqrt{\gamma}\, M_0 \dot{X}_i \tag{2}$$

is a gauge dependent quantity. The quantity $\sqrt{\gamma}\, d\sigma$ is the measure of the string's proper length. The vortex momentum consists of two terms, of which the gauge dependent first one, P_i^{inc}, is the contribution of the vortex-velocity dependent term in the Magnus action S_M (it corresponds to the electrodynamical minimal coupling term qa_i). The second one, the kinetic (vortex matter) part $P_i^{\text{kin}} = \sqrt{\gamma}\, M_0 \dot{X}_i$, comes into play if the finite compressibility of the superfluid is taken into account. This hydrodynamic contribution to the vortex momentum is acquired from the self interaction of the vortex string with phonons. The static mass of the vortex is in the present non-charged case $M_0 = E_{\text{self}}/c_s^2$, *i.e.* its static self energy E_{self} divided by the speed of sound squared [19]. In the given form the static mass is of relevance in the actual dense superfluid if vortex velocities remain semiclassical, that is, much less than the Landau critical velocity related to the quantum many-body structure of the superfluid. The hydrodynamic mass is, up to a factor of order unity, given by $M_0 \sim M_{\text{core}} \ln[R/(\xi e^{C_0})]$, where R is the infrared cut-off (distance to the next vortex or size of the vessel) and C_0 is a core constant, reflecting our ignorance of the core structure. The core mass $M_{\text{core}} = m\rho_0\pi\xi^2$ is a measure of the mass per unit length of the normal fluid contained in the core.

The contribution of $\boldsymbol{P}^{\text{kin}}$ (coming mainly from region II in Fig. 1) is very small compared to that of the Magnus force acting in region I and II [14–17]. The

reason lies in the fact that the ratio of these two quantities,

$$\frac{\left|\boldsymbol{P}^{\mathrm{kin}}\right|}{\left|\boldsymbol{P}^{\mathrm{inc}}\right|} \approx \frac{\kappa}{|\boldsymbol{X}|c_s}\frac{|\dot{\boldsymbol{X}}|}{c_s}\,, \tag{3}$$

depends on $|\dot{\boldsymbol{X}}|/c_s$ as well as $\kappa/(c_s|\boldsymbol{X}|)$ ($=$ O$(\xi/|\boldsymbol{X}|)$ in helium II). Both quantities are necessarily $\ll 1$ if the vortex is to be described semiclassically. We used here that $|b_{ij}|$ is of order $|X^k|$ in a Cartesian frame, and neglected the dependence of the self energy logarithm on the vortex co-ordinates.

The vortex action has the canonical form

$$S_V = \int_0^T\!\oint dt\, d\sigma\, \dot{\boldsymbol{X}}\cdot\boldsymbol{P} - \int_0^T dt\, H_V\,, \tag{4}$$

where the Hamiltonian H_V is the sum of static, kinetic, elastic and potential terms [16]:

$$H_V = \oint d\sigma\sqrt{\gamma}\left[M_0 c_s^2 + \frac{1}{2\gamma M_0}\left(\boldsymbol{P}-\boldsymbol{P}^{\mathrm{inc}}\right)^2 \right.$$
$$\left. + \frac{M_0 c_s^2}{2\gamma}\,\boldsymbol{Q}'^2\right] - m\rho_0\kappa\int d\sigma\left(\frac{1}{2}\boldsymbol{\psi}_C + \boldsymbol{\psi}_u\right)\cdot\boldsymbol{X}'\,. \tag{5}$$

Here, the vector $\boldsymbol{Q}$ is lying in the plane locally perpendicular to $\boldsymbol{X}'$ and measures deviations from an equilibrium shape of the string. The stream function vector $\boldsymbol{\psi}$ (see Appendix A) is separated into its Coulomb part $\boldsymbol{\psi}_C$ from the interaction of the vortex in question with other vortices and the part $\boldsymbol{\psi}_u$ stemming from the interaction with a (constant) background flow field. The first term in the square brackets represents the static, the second the kinetic, and the last one the elastic energy of the string. In this representation for H_V, we imposed the Coulomb gauge condition div$\boldsymbol{P}^{\mathrm{inc}} = 0$, which results in the factor of 1/2 in front of the 'Coulomb' potential $\boldsymbol{\psi}_C$.

3.2 Co-ordinates and momenta

The rotational part of $\boldsymbol{P}^{\mathrm{inc}}$ satisfies rot $\boldsymbol{P}^{\mathrm{inc}} = -m\rho_0\kappa\boldsymbol{X}'$ (this follows from (A.4)). Using a local righthanded co-ordinate basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_\sigma \equiv \boldsymbol{X}'$ with

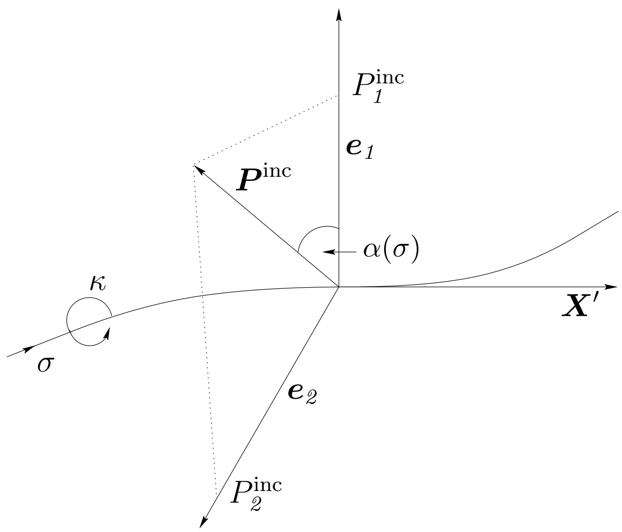

Fig. 2. The momentum $\boldsymbol{P}^{\mathrm{inc}}$ defined from the Magnus force action (1) is a gauge dependent quantity. On every point σ of the vortex line it can point in an arbitrary direction of the local co-ordinate plane $\boldsymbol{e}_1$, $\boldsymbol{e}_2$. This direction is parameterized by the angle $\alpha(\sigma)$.

determinant g, this yields the gauge invariant relation [17]

$$\partial_2 P_1^{\mathrm{inc}} - \partial_1 P_2^{\mathrm{inc}} = m\rho_0\kappa\sqrt{g}\,. \tag{6}$$

This equation expresses the well-known conjugateness of positions and momenta for a vortex line in an incompressible fluid (see, *e.g.*, [20]). It appears here in its generalized form, valid for any choice of the co-ordinate basis on the string. Co-ordinates and momenta in different co-ordinate directions are no longer independent, just as for the analogous electrically charged particle in a very large magnetic field. The momentum direction depends on the local co-ordinate basis as well as on the particular gauge chosen. The Coulomb gauge for H_V in (5) is just one possibility. A convenient and often used gauge is to simply set $P_2^{\mathrm{inc}} = 0$ so that $P_1^{\mathrm{inc}} = m\rho_0\kappa \int dX^2 \sqrt{g}\,(\boldsymbol{X})$. The co-ordinates and momenta can functionally depend on each other in a complicated, non-linear fashion, according to the choice of the co-ordinate system, which can be local in σ. Of importance in the context treated, though, is the fact that all the local momenta are given in terms of the local co-ordinates.

A particularly well-known example [20] for conjugateness of the co-ordinates is provided by the Cartesian co-ordinates X, Y of a rectilinear line in z direction. In the gauge $P_Y = 0$, $P_X = m\rho_0\kappa Y = h\rho_0 Y$ and the commutator is

$$[\hat{X}, \hat{Y}] = i(2\pi\rho_0^{(2)})^{-1}\,, \tag{7}$$

where $\rho_0^{(2)}$ is the 2d bulk number density in the x-y plane. For a circular vortex in the gauge $P_R = 0$ (cylindrical co-ordinates, $\sqrt{g}(\boldsymbol{X}) = R$), the canonical momentum per unit length of the azimuthal co-ordinate is $P_Z(\sigma) = (1/2)m\rho_0\kappa R^2 = (1/2)h\rho_0 R^2$ and we have

$$[\hat{Z}, \hat{S}] = i(2\pi\rho_0^{(3)})^{-1}\,, \tag{8}$$

where $\hat{S}$ is the operator of the surface area $S = \pi R^2$ of the vortex ring.

3.3 Volume elements in the tunnelling exponent

The quantity of relevance for the description of tunnelling with constant energy E is the Legendre transform $S_e(E)$ of the Euclidean action $S_e(T)$ (T is the period of motion in Euclidean time). The Euclidean action as a function of imaginary time is defined from the real time action by $S_e = -iS[t \to -it]$.

$$\begin{aligned} S_e(E) &= S_e(T) - \frac{\partial S_e}{\partial T} T = S_e(T) - ET \\ &= -i \oint d\sigma \oint d\boldsymbol{X} \cdot \boldsymbol{P} = \oint d\sigma \oint d\boldsymbol{K} \cdot \boldsymbol{P}\,, \end{aligned} \tag{9}$$

where we defined the imaginary co-ordinate $\boldsymbol{K} = -i\boldsymbol{X}$ of the vortex [2]. The meaning of $S_e(E)$ is most easily understood if we realize that the function $\exp[-S_e(E)/\hbar]$ is an energy dependent damping factor for a quantum mechanical wave function, traveling in imaginary time (with imaginary momentum), and penetrating a potential barrier (cf., in particular, the textbook [21] for a proper treatment of the semiclassical approximation).

The peculiar property of a massless vortex, as seen in the last subsection, is provided by the fact that all the momenta have to be functions of the co-ordinates. If we forget about the small correction of the kinetic part to the momentum and set $\boldsymbol{P} = \boldsymbol{P}^{\rm inc}$, using (6), we can thus bring the Euclideanized vortex action in units of Planck's quantum of action into the form [3] [17]:

$$\frac{S_e(E)}{h} = \rho_0 \iiint \sqrt{g}\, d\sigma\, dZ^1 dZ^2 = \rho_0 \Omega^{(d)} \tag{10}$$

The co-ordinate differentials are represented by $dZ^1 = \cos\alpha\, dK^{1} + \sin\alpha\, dK^{2}$, $dZ^2 = -\sin\alpha\, dX^{1} + \cos\alpha\, dX^{2}$. The angle $\alpha(\sigma)$ is the parameter giving the lo-

[2] $\boldsymbol{K}$ is not to be confused with a wave vector. We could have chosen as well to incorporate the $-i$ into the momentum $\boldsymbol{P}$. Crucial is only that S_e is a real quantity.

[3] We omit the subscript V for the Euclidean action.

cal direction of the canonical momentum to be chosen subject to the constraint (6). The quantum of circulation was taken to be $\kappa = h/m$. More generally, $\kappa = (N_v/N_s)h/m$, with N_v the vortex topological winding number and N_s the number of particles with mass m in the elementary boson.

That the only variables in the problem are effectively co-ordinates yields an expression for the tunnelling exponent which is in principle very simple. Given that we can solve for the motion of the local volume element on the line $\sqrt{g}\, d\sigma\, dZ^1 dZ^2$ along the whole path of the line in co-ordinate space as a function of t, σ, and its motion stays completely semiclassical, we can calculate the tunnelling exponent of the vortex to arbitrary precision. The motion of the volume element in (complex) configuration space (which represents simultaneously the phase space) determines completely the semiclassical tunnelling exponent in the incompressible limit, that is, if this exponent is dominated by the contributions of region I in Fig. 1. This is always the case if $S_e(E) \gg \hbar$. We will leave this semiclassical domain if the dimensionless action is approaching some number of order one or if characteristic variations of the co-ordinates take place on scales of the order of the core size ξ.

The integral (10) expresses the Bohr-Sommerfeld quantization of the number of particles $N^{(d)}$ contained in the tunnelling volume $\Omega^{(d)}$:

$$S_e(E) = (N^{(d)} + \gamma)h \quad \Leftrightarrow \quad S_e(E)/\hbar = 2\pi(N^{(d)} + \gamma). \tag{11}$$

The number γ is of the order one and signifies the onset of the microscopic quantum regime. In the semiclassical approximation, $N^{(d)} \gg \gamma$ must hold.

3.4 Violation of Galilean Invariance

At the absolute zero of temperature, a homogeneous non-relativistic superfluid has Galilean invariance, that is, physical contents are invariant under co-ordinate transformations to any reference frame at constant velocity. If we approach absolute zero, which is what is actually realized in experiment, we expect the tunnelling rate to make no abrupt change as the temperature is lowered. Thus the result for the rate we obtain at $T = 0$ should also be valid for temperatures slightly above $T = 0$.

Because we can always transform to the rest frame comoving with the superfluid, the tunnelling probability at $T = 0$ equals zero if Galilean invariance remains unbroken: In the rest frame there is a tunnelling barrier of infinite height, the logarithmically diverging vortex self energy. Hence it is necessary to explicitly include the violation of Galilean invariance by a flow obstacle into any calculation of tunnelling rates for Galilean invariant superfluids at

absolute zero.

The most likely location for the breakdown of invariance to happen in a pure superfluid is the boundary of the superfluid having some surface roughness extending on scales much larger than ξ. We will describe in the section which follows the simplest possibility to explicitly solve for the motion of a vortex in a boundary geometry of varying curvature. This will provide us with a picture of the geometrical restrictions on vortex quantum tunnelling in the semiclassical domain.

4 Geometry Dependence of Vortex Quantum tunnelling

It is obvious that to solve in general for the string motion of constant energy, according to the equations of motion resulting from (4) and the Hamiltonian (5), is a quite complicated task in some nontrivial boundary geometry. Determining the tunnelling volume $\Omega^{(d)}$ will be difficult in three spatial dimensions if we do not assume, like in [2], the highest possible symmetry of a undeformed, massless ring vortex in the presence of a sphere. Even if we neglect any kinetic and elastic terms of vortex motion, the problem will require a quite formidable computational effort because to solve it, *e.g.*, by the image technique will require a multitude of image vortices.

We will thus cut the problem down on two dimensions to display the general properties and dependencies of the tunnelling volume on external geometry. This of course, then, does not describe any possible influence of waves traveling along the line. Their contribution to the Euclidean action, however, will be suppressed comparable to that of the kinetic term in (5), as long as semiclassicality is retained.

4.1 Analytical hydrodynamic solution in two spatial dimensions

The basic solution from which we start is that for a vortex in the presence of a half-circle of radius d at an otherwise flat boundary (cf. Fig. 3). The complex plane of this original solution is called the Z-plane. The complex potential [18] generated by the image vortices and acting on the vortex at Z_1 is given by

$$w_i(Z_1) = -i\,\frac{\kappa}{2\pi}\ln\left[\frac{\left(Z_1-\bar{Z}_1\right)\left(d^2/Z_1-\bar{Z}_1\right)}{d^2/Z_1-Z_1}\right]. \tag{12}$$

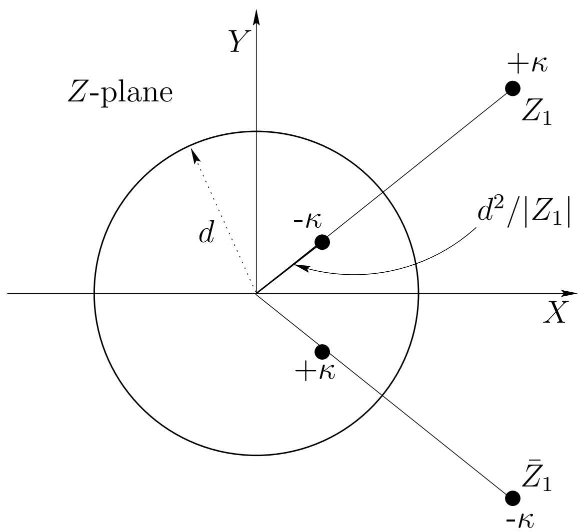

Fig. 3. The most simple nontrivial boundary problem solvable by the image technique: A point vortex in the half space $Y > 0$, which is filled with liquid, moving near a (half-)circle. The boundary conditions are satisfied by an image vortex at the plane and two image vortices of opposite strength inside the circle.

There a three contributions, coming from the three image vortices depicted in Fig. 3. The first factor in the numerator stems from the image vortex at the plane $Y = 0$ with complex potential $w(Z) = -i(\kappa/2\pi)\ln[Z - \bar{Z}_1]$, the second one is obtained by the circle theorem [18] as the image of the original vortex at the circle. Finally, the potential of the remaining $+\kappa$-circulation vortex inside the circle, contributing in the denominator of the logarithm, completes the image vortex system, again by the circle theorem.

The imaginary part of the complex potential gives the stream function $\psi = \Im[w]$, whereas the real part is the usual velocity potential. The first term in the denominator of the logarithm is incorporated into the static self energy of the vortex, $E_{\rm self} = (m\rho_0\kappa^2/4\pi)\ln\left(|Z_1 - \bar{Z}_1|/\xi\right)$, which is cut off by ξ and equal to half the energy of a vortex pair separated by $|Z_1 - \bar{Z}_1|$. The expression for the potential in (5) is thus $\psi_C = -(\kappa/2\pi)\ln\left(\left|(d^2/Z_1 - \bar{Z}_1)/(d^2/Z_1 - Z_1)\right|\right)$.

In addition, we superimpose a constant external superflow of velocity u from right to left. This flow has complex potential

$$w_u(Z) = u\left(Z + \frac{d^2}{Z}\right) . \tag{13}$$

From (5), we can then infer the energy of the point vortex:

$$\tilde{E}(Z_1) = \ln\left|\frac{(Z_1 - \bar{Z}_1)\left(d^2/Z_1 - \bar{Z}_1\right)}{\xi\left(d^2/Z_1 - Z_1\right)}\right|$$

$$-\frac{4\pi u}{\kappa}\Im\left(Z_1+\frac{d^2}{Z_1}\right)\,. \tag{14}$$

The energy is normalized by $m\rho_0\kappa^2/4\pi$, the characteristic energy of the 'particle' vortex (corresponding in the electrostatic analogy to the charge squared divided by 4π [16]). We omitted in the above equation the contribution of vortex inertia, $(1/2)\tilde{M}_0|\dot{Z}_1|^2$ (where $\tilde{M}_0=\tilde{E}_{\rm self}/c_s^2$), *i.e.* wrote down the energy in the incompressible limit.

We wish to map by a conformal transformation the region outside a boundary surface with varying curvature radius, lying in the z-plane, to the domain outside the circle. Any such transformation can be written $z=a_0Z+\sum_{n=0}^{\infty}b_nZ^{-n}$, where a_0,b_n are some coefficients and $Z=d\exp(i\chi)$ is on the circle [18]. We would like to invert this relation to obtain the solution for the boundary surface directly from that for the circle. The simplest possibility to do so is supplied by choosing $b_0=0$, $b_n=0$ for $n>1$ and scaling a_0 to unity, which leads to the celebrated *Joukowski Transformation*

$$z=Z-l^2/4Z\,, \tag{15}$$

which maps the outside of the ellipse with half axes a,b ($a<b$) to the outside of the circle of radius $d=(a+b)/2$. The parameter l is defined by $l^2=b^2-a^2$. The inversion of the transformation (15) $2Z=z+\sqrt{z^2+l^2}$ gives, upon insertion into the solution for the circle, the expression for the vortex energy in the presence of the ellipse. It can be seen from the structure of the inverse transformation that the use of elliptic co-ordinates, defined by $z=l\sinh\zeta$, is most convenient. Here, $\zeta=\chi+i\eta$ is the new complex co-ordinate, related to a Cartesian system by $x=l\sinh\chi\cos\eta$, $y=l\cosh\chi\sin\eta$. The co-ordinate lines are confocal ellipses and hyperbolas, and $2Z=l\exp(\zeta)$.

In what follows, we consider the solution (14) in the half-plane $y>0$. The normalized energy as a function of the elliptic vortex co-ordinates χ_1,η_1 takes the form

$$\begin{aligned}\tilde{E}(\chi_1,\eta_1)=\ln&\left[\frac{a+b}{\xi}\,\frac{\exp(\chi_1-\chi_0)|\sin\eta_1|\sinh(\chi_1-\chi_0)}{(\sinh^2(\chi_1-\chi_0)+\sin^2\eta_1)^{1/2}}\right]\\&-\frac{4\pi u(a+b)}{\kappa}\sinh(\chi_1-\chi_0)|\sin\eta_1|\,,\end{aligned} \tag{16}$$

where $\chi_0=\mathrm{artanh}(a/b)$ is the co-ordinate specifying the ellipse surface. The shape of the energy in the χ-η plane is shown in Fig. 4.

The velocity u will henceforth be scaled with the characteristic velocity $v_L=$

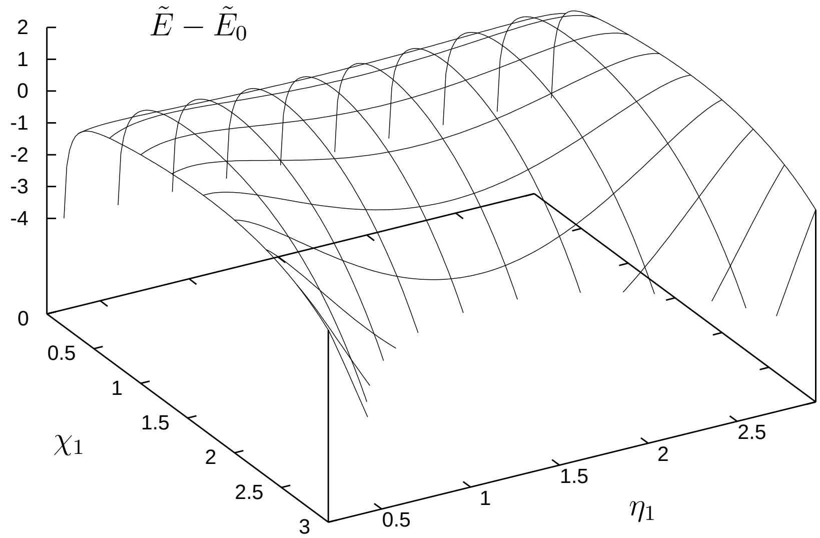

Fig. 4. Shape of the potential barrier (16) with the choice $\chi_0 = 0.175$. The corresponding ellipse with $b/a \simeq 5.7$ is shown in Fig. 5. The (large) velocity $u = 0.08$ is in units of $v_L = \kappa/2\pi\xi$ and the ratios $a/\xi = 2$, $s/\xi = 1$. The zero of this normalised potential energy is shifted by $\tilde{E}_0$, defined in (17).

$\kappa/2\pi\xi$, expressing a *velocity limit* for the validity of the semiclassical approximation. (The Landau critical velocity of roton creation $\simeq 59$ m/s at $p \simeq 1$ bar equals v_L if $\xi \simeq 2.7$ Å.) The velocity $u = 0.08$ chosen for the potential in Fig. 4 is large and near critical in the sense that the local velocity at the ellipse top is near v_L. It appears from this graph in real co-ordinate space that the vortex will predominantly tunnel in the χ-direction around the ellipse top ($\eta \approx \pi/2$), because there the potential barrier is shallowest.

4.2 Geometric restrictions for semiclassicality

Of course it is possible to invent more intricate conformal transformations to map the circle solution onto fancy shapes in the z-plane. But the ellipse already possesses the feature crucial for an analysis of vortex motion in a geometry deviating from the highest symmetry of Fig. 3: It has curvature varying continuously between a/b^2 in the x- and b/a^2 in the y-direction. This leads to a restriction on the possible vortex paths near the boundary [17], visualized in Fig. 5.

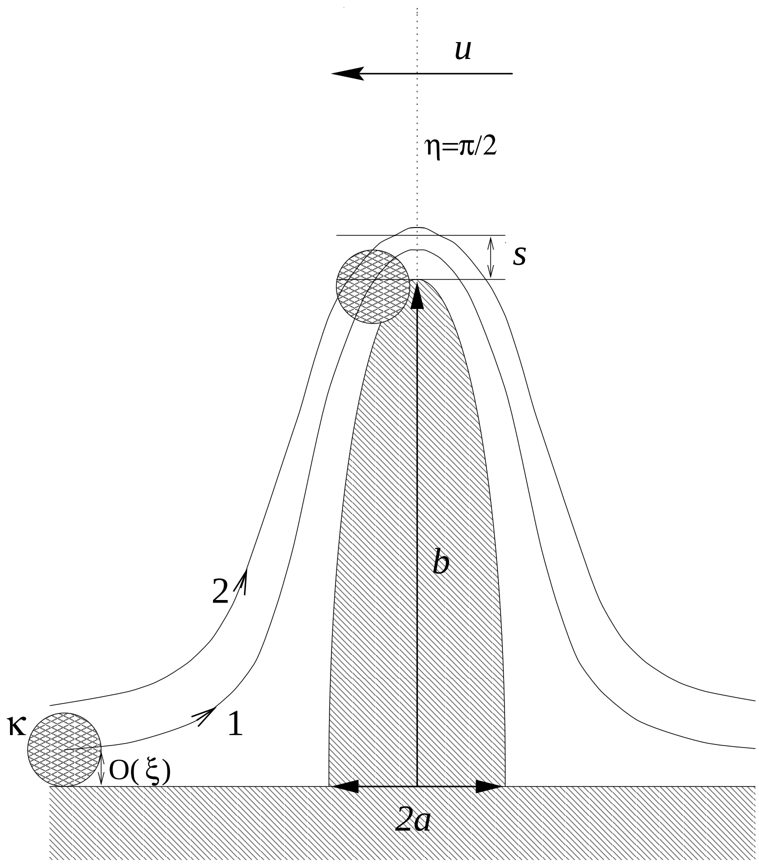

Fig. 5. Two vortex paths of constant energy near the ellipse. Whereas the vortex on path 1 with approximately zero energy, $\tilde{E} \simeq 0$, does not manage to pass by without coming closer than ξ, the second one, having energy $\tilde{E} = \tilde{E}_0$, defined in (17), is able to do so. The velocity u is to be understood 'at infinity'.

On the first path we consider a vortex having approximately zero energy $\tilde{E} \simeq 0$, starting far away from the ellipse with a distance of the order ξ to the boundary. As it approaches the ellipse and tries to get around it, it will inevitably hit the boundary because it does not have enough energy to cross the sharp ellipse top at a safe distance. 'Safe', that is, such that the semiclassical approximation can be applied. For that purpose, the vortex has to have a distance to the boundary exceeding the many-body core size ξ. This imprints a restriction on the value of the possible tunnelling energies for a geometry in which the curvature radius decreases (the curvature increases) while the vortex is passing by: The energy has to be chosen with a value at least high enough such that the vortex is able to pass the complete boundary at a distance larger

than ξ. Introducing the *closest approach distance* s to the boundary (see Fig. 5), and observing that for $\delta\chi = \chi_1 - \chi_0 \ll 1$ the distance of the vortex to the ellipse top is given by $\delta b \simeq a\delta\chi$, we impose $\delta\chi(\eta_1 = \pi/2) = s/a$, where we assume $b \gg a \gg s$ and $s \geq \mathrm{O}(\xi)$.[4] Under these conditions, the vortex center on path 1 in Fig. 5 will pass the top at the distance $\delta b \simeq (a/b)\xi$, whereas the path 2 vortex with energy $\tilde{E}_0 = \tilde{E}_0(a, b, s, \xi, u)$,

$$\begin{aligned}\tilde{E}_0 &= \ln\left[\frac{a+b}{\xi}\exp\delta\chi\,\tanh\delta\chi\right] - \frac{4\pi u(a+b)}{\kappa}\sinh\delta\chi \\ &\simeq \ln\left[\left(1+\frac{b}{a}\right)\frac{s}{\xi}\right] - \frac{2u}{v_L}\left(1+\frac{b}{a}\right)\frac{s}{\xi},\end{aligned} \tag{17}$$

passes at $\delta b \simeq s$. The energy $\tilde{E}_0$ is the energy needed by the vortex to remain completely describable in semiclassical terms on its way along the ellipse. In general, it depends on u, but in the low velocity limit $2u/v_L \ll 1$, it is given by $\tilde{E}_0 \simeq \ln[(b/a)(s/\xi)]$. The effect of this nonzero energy can be interpreted as the rescaling of $\xi \to \xi \exp \tilde{E}_0$ in (16). The energy is zero *with respect to* this rescaled ξ. For small u, the core size rescales as $\xi \to (b/a)s$.

That the validity of the semiclassical approach enforces that we introduce another geometrical quantity, s, is a restriction of quite general character. It is of relevance for any attempt to describe tunnelling semiclassically in a realistic, non-spherical geometry, *i.e.* when the boundary and thus the path of the tunnelling object near it is not of S^n symmetry. A semiclassical description is valid only if the quantum core structure of the tunnelling object is not touched upon during its motion along the boundary. A pinning potential for the vortex moving in the superfluid stems in general from some flow obstacle, in our case the ellipse. Any phenomenological ansatz for a pinning potential usually employed in tunnelling calculations which has curvature perpendicular to the applied flow larger than parallel to the flow will have to take into account that the object can approach the surface within its core size, invalidating the collective co-ordinate description.

It is to be noted that the energy is in units of $m\rho_0\kappa^2/4\pi \simeq 0.82$ K/Å (at $p \simeq 1$ bar and in three dimensions). For realistic values of the parameters in (17), the values of $\tilde{E}_0$ cover the same range as the phonon-maxon-roton spectrum. Energywise, the trapped vortex thus can not be distinguished from an elementary excitation. It could have been excited thermally and remained trapped

[4] In the relation $s \geq \mathrm{O}(\xi)$, the quantity $\mathrm{O}(\xi)$ means 'a value in the order of ξ by definition', as there is no sharp distinction between the 'inside' and 'outside' of a quantum vortex. Additionally, the choice for the lower limit value of the distance s in units of ξ depends on the value of the core constant $C_0 = \mathrm{O}(1)$, parameterizing the many-body core structure in the vortex energy logarithm $\ln[R/(\xi \exp C_0)]$ ($\exp C_0 = 1$ for the point vortex considered, which has $R = 2Y$).

in a pinning center during the cool-down of the superfluid to very low temperatures. From this and the above analysis it follows that it is semantically not appropriate to use the term ‘nucleation’ if we remain completely semiclassical and define it to mean ‘creation from nothing pre-existent’, that is, from the zero of energy. Experimentally, it will be impossible to distinguish the tunnelling of a pre-existing small energy vortex at a rough boundary from the true nucleation event of a nascent vortex there, if no direct means to control the microscopic dynamics can be provided.

4.3 The tunnelling area

In the effectively one-dimensional problem we are considering (X is the location of the vortex, $P_X = m\rho_0\kappa Y = h\rho_0 Y$ its momentum), there always exists a closed vortex path in phase space. For the tunnelling trajectory this path is in the K-Y plane and the closed vortex path is running underneath the barrier of Fig. 4. It has in the limit of small velocities the shape shown in Fig. 6.

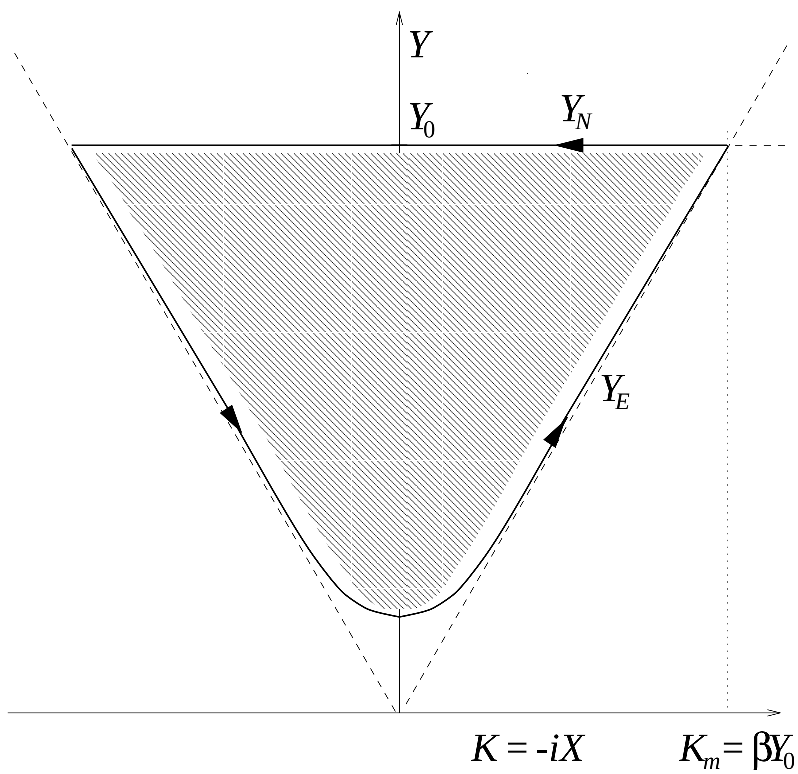

Fig. 6. The closed vortex path giving the action (18) in the low velocity limit. The first part Y_E corresponds to the analytically continued path 2 of Fig. 5 along the ellipse surface. The second part of the closed path, $Y_N \simeq Y_0$, obtained from the second solution of (16) for the constant energy in (17), represents the border line to a free vortex.

The action (9) in the form of (10), which follows from the closed vortex path in the low velocity limit [17],

$$\frac{S_e}{h} = 2\rho_0 \int_0^{K_m} (Y_{\rm N} - Y_{\rm E}) dK \simeq \rho_0 \beta Y_0^2$$
$$= \rho_0 \left(\frac{a}{b} + \frac{s}{a}\right) \left(\frac{\kappa}{4\pi u} \ln\left[\frac{\kappa}{2\pi u}\frac{a/b}{s}\right]\right)^2 = \rho_0 \Omega^{(2)} , \quad (18)$$

is bounded from below by the fact that the tunnelling area $\Omega^{(2)}$ can not be arbitrarily reduced without violating semiclassicality, *i.e.* by the introduction of s. Geometrically, the curvature radius of the vortex path near the flow obstacle can not be made arbitrarily small to fit a given shape of the asperity on the boundary, so that the eccentricity $\epsilon \simeq 1 - 0.5\beta^2$ of the path can not approach unity arbitrarily close. The bigger of the two quantities $a/b, s/a \geq \xi/a$ decides about the lower limit of $\Omega^{(2)}$, attainable in semiclassical tunnelling.

The *materialization distance* Y_0 of the vortex is in the low velocity limit given by (in units convenient for the discussion of experiments which follows below):

$$Y_0 \simeq \frac{0.8\,{\rm nm}}{u\,[10\,{\rm m/sec}]} \ln\left(\frac{v_L}{u}\frac{a/b}{s/\xi}\right) . \quad (19)$$

In three dimensions, the radius of the materialized vortex half-ring, $R_0 = (\kappa/2\pi u)\ln(\cdots) = (1.6\,{\rm nm}/u[10\,{\rm m/sec}])\ln(\cdots)$, bears an additional factor of 2 in front of the logarithm.

5 Prefactor estimations; Relation to experimental data

The tunnelling probability has the form $P(E) = A\exp[-S_e(E)/\hbar]$, where A is the so-called prefactor depending on the fluctuations of the quantum variables around their classical value. Apart from the considerable difficulties in evaluating prefactors in general, an accurate calculation of A in a dense superfluid like helium II is in principle not possible due to the lack of a microscopic theory. It is, however, feasible to get an idea about the value of this prefactor within about two orders of magnitude. This is all what we need, because, as we will see below, the variations of the tunnelling exponent $S_e(E) \gg \hbar$ with the (geometrical) parameters of the problem will dominate anything which is actually observable (provided of course that a semiclassical, large scale description is appropriate). The simplest possible idea about the prefactor is gained by considering the frequency ω_a of a particle oscillating in a metastable well. Then, neglecting the influence of dissipation on the vortex motion, within about one

order of magnitude $A \sim \omega_a$ [27]. In the thermal activation limit, *i.e.* in the Arrhenius law case $P = (\omega_a/2\pi)\exp[-U/k_BT]$, ω_a is the frequency of oscillations in the metastable well against a barrier of height U. The frequency $\nu_a = \omega_a/2\pi$ can thus be generally understood as a measure of the number of times per second the vortex bounces against the potential barrier, trying to get free. We have no possibility to describe the vortex state (at the boundary) quantitatively, but we are able to conclude on the order of magnitude of ω_a by taking into account that there exists a surface layer of vorticity of width ξ. Because the superfluid density goes to zero at the boundary and heals back within ξ, the energy needed for the activation of vortices vanishes within this distance [3]. The frequency of motion of these vortices should then be of order

$$\omega_0 = \frac{\kappa}{\pi\xi^2} = 4.87 \cdot 10^{11}\mathrm{sec}^{-1}\left(\frac{\sigma_{\mathrm{LJ}}}{\xi}\right)^2 , \qquad (20)$$

which is the cyclotron frequency of vortex motion. We scaled ξ with the Lennard-Jones parameter $\sigma_{\mathrm{LJ}} = 2.556$ Å of the ^{4}He atomic interaction. Approximately the same estimate is obtained if we directly take into account compressibility in the form of a finite speed of sound. The rigidity (spring constant) of the vortex against deformations will scale as $1/\xi^2$, its mass as $1/c_s^2$, so that

$$\omega_s = \frac{c_s}{\xi} \simeq 9.23 \cdot 10^{11}\mathrm{sec}^{-1}\frac{\sigma_{\mathrm{LJ}}}{\xi} , \qquad (21)$$

where we took the pressure to be $p \simeq 1$ bar. It should be stressed that the approximate coincidence of the estimates (20) and (21) is a particular feature of helium II, where $\kappa \approx 2\xi c_s$. If we adhere to the description of a massive vortex of size ξ moving in some (regularized) potential, the equation (21) is to be used, whereas (20) is the natural vortex frequency associated with the scale ξ alone.

Assuming that the prefactor can vary between $A \approx 10^{10} \cdots 10^{12}$ Hz, its logarithm $\ln A \simeq 23 \cdots 28$. It is thus obvious that variations of the action in the tunnelling exponent with geometrical factors will outweigh by far any variations in the prefactor. The tunnelling exponent in (11) involves N, the number of particles in the tunnelling volume. Corresponding to the conceivable values of the prefactor, it will have to vary between $N = 4 \cdots 6$, say, for tunnelling events to be observable within a reasonable span of experimental time. Though this number is small, it is not impossibly so for the hydrodynamic treatment to make sense at least asymptotically (*asymptotic* to the microscopic nucleation regime).

The experiments in which one hopes to observe intrinsic[5] vortex quantum tunnelling at irregular boundaries [22] are using flow orifices of sub-micron size [23]. An oscillating superflow of frequency in the order of Hz is driven through the orifice. It is observed that at a certain critical value of the amplitude of the diaphragm driving the flow through the hole, there is an instantaneous (on the scale of the driving frequency) breakdown of the amplitude which is quantized [24]. This quantized dissipation event is associated with a vortex generated at the orifice walls, subsequently crossing all the streamlines of the flow, thereby causing a phase slip event [25], which draws a quantized amount of energy from the flow. The critical velocity, at which the phase slip takes place, is first linearly increasing with decreasing temperature and then saturates at $T \simeq 150$ mK to a plateau (*flat* within experimental resolution). This plateau has been conventionally associated with the quantum tunnelling of half-ring vortices through a potential barrier [26].

In the experiments performed so far, the mean critical velocity through the orifice on the plateau is of the order 5-10 m/s. Though this is not necessarily equal to the velocity u at 'infinity' (think, *e.g.*, of a sharp spike on a smoother asperity at the orifice wall), the radius R_0 of the materialized vortex half-ring can be estimated to be in the order of nanometers. This should be sufficiently big for the hydrodynamic approach to make sense. The crossover temperature T_0 from thermal activation to quantum tunnelling is, for negligible dissipation and small u, given by $2\pi k_B T_0 = \hbar\omega_b$, where ω_b is the oscillation frequency of vortex motion in the inverted potential (see, *e.g.*, [27],[28]), which yields $\omega_a \simeq \omega_b = 8.2 \cdot 10^{10} T_0$ [100 mK] Hz. This is about an order of magnitude smaller than the estimates (20) respectively (21). There are, however, several facts making a direct comparison of these values questionable. First of all, these estimates can give only a rough idea about the true dynamical behaviour of a many-body vortex near the boundary. It is conceivable, for example, that the effective 'spring constant' of the vortex against deformations is lowered compared to the semiclassical estimate in (21) because of the many-body quantum uncertainty of its position. In addition, we have seen in the preceding section that a vortex will in general not be in a state with zero energy at the wall. It will rather be in a state with the energy $\tilde{E}_0$ in (17). This can also cause a change in the prefactor. Furthermore, the prefactor is in general a function of driving velocity u and temperature T [28] and at the measured crossover temperature in terms of the critical velocity not necessarily equal to its value at zero temperature. This value will presumably be closer to the estimates (20) and (21).

[5] The term *intrinsic* is meant to be the opposite of 'supported and supplied by pre-existing vortices of macroscopic size', *i.e. extrinsic.*

6 Conclusion

There is no theory of vortex quantum nucleation in the dense Bose superfluid helium II which could resolve the problems connected with theoretical issues and experimental facts. There is, though, a consistent theory of vortex quantum tunnelling in the semiclassical domain, which we represented here in its formal requirements and geometric implications. It is again to be emphasised that this theory does not make any assumptions about the dynamical behaviour of the superfluid on scales approaching the microscopic one. This implies on one hand that it necessarily can not supply a theory of vortex nucleation. On the other hand, it gives semiclassical bounds on tunnelling rates which can be experimentally observed. It should have become apparent from our discussion that to measure the consequences of the actual microscopic nucleation event and, in particular, influence the (microscopic) parameters which govern its probability, will be a task quite difficult to realize. If the tunnelling event takes place on sufficiently large semiclassical length and small velocity scales, *i.e.* is sufficiently semiclassical, the only remnant of microscopic dynamics we can expect to be involved in the tunnelling rate is the core size ξ. But this is only appearing through a logarithm, which means that its influence on the tunnelling rate is quite subordinate as compared to the tunnelling exponent variation with the path geometry of the vortex, imprinted on it by the flow obstacle.

The very fact of quantum tunnelling at boundaries needs further proof so that the predictions of tunnelling theory can be compared to that of classical instability mechanisms (investigations in this direction are found under reference [29]). One such proof could consist in the comparison of critical velocities for chemically identical orifices of equal global sizes, which have different microscopic surface structures. If the result of such measurements is negative, *i.e.* there is no reproducible difference in critical velocities, there is no quantum process taking place describable by (semiclassical, hydrodynamic) means of tunnelling at irregular boundaries. The experimental information at present is too sparse to give a final and conclusive answer.

Acknowledgement

I would like to thank Grisha Volovik and Kolya Kopnin for discussions and the Low Temperature Laboratory of the Helsinki University of Technology, where this work has been done, for hospitality. Financial support of this research was provided by the Human Capital and Mobility Programme of the European Union under grant CHGE-CT94-0069.

Appendix

A The dual formulation

In this appendix, we introduce the reader to the basics of the dual formulation of string dynamics. For ease of reading the formulas are given in the conventional index notation, which will be familiar to the majority of the readership, as well as in a co-ordinate independent formulation using p-forms, which shows most clearly their geometrical meaning [30]. We give the covariant formulae first, subsequently reducing them to their Galilean limits, which are of relevance in superfluids like helium II.

In the hydrodynamic limit, a neutral superfluid of spinless bosons is described by an order parameter function $\phi = \rho^{1/2}(x)\exp[i\theta(x)]$ of the spacetime point $x = (t, \vec{x})$ (the velocity of light $c \equiv 1$). Its absolute square is the density $\rho(x)$ and its phase $\theta(x)$ the velocity potential of the fluid. The fundamental conservation law derived from this semiclassical U(1)-description of the superfluid is that of the hydrodynamic mass four-current $\boldsymbol{J} \equiv m\rho\boldsymbol{v} = \hbar\rho\, \boldsymbol{d}\theta = \hbar\rho\, \partial_\mu\theta\, \boldsymbol{d}x^\mu$, where $\partial_\mu \equiv \partial/\partial x^\mu$. The most familiar mathematical form of mass conservation is provided by $\partial_t\rho + \mathrm{div}(\rho\vec{v}) = 0$. This is expressed covariantly as

$$\nabla_\mu J^\mu = 0 \quad \Leftrightarrow \quad \boldsymbol{d}\wedge{}^*\boldsymbol{J} = 0 \qquad (\boldsymbol{d} \equiv \boldsymbol{d}x^\alpha\partial_\alpha)\,. \tag{A.1}$$

That is, the dual of the current $\boldsymbol{J} = J_\mu \boldsymbol{d}x^\mu$ is a 3-form ${}^*\boldsymbol{J} = J^\mu\epsilon_{\mu\nu\alpha\beta}\, \boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\alpha \wedge \boldsymbol{d}x^\beta$, which is closed. This statement is equivalent to a vanishing covariant derivative $\nabla_\mu J^\mu$ of the current vector with components J^μ. We define the *field strength* $\boldsymbol{H}$ by ${}^*\boldsymbol{J} = m\rho_0\boldsymbol{H}$, where ρ_0 is the bulk constant density. The field $H_{\mu\nu\alpha}$ is totally antisymmetric in its three indices and has, by definition, only four independent components. In a simply connected region, $\boldsymbol{H}$ is exact, *i.e.* it is the exterior derivative of a *gauge* 2-form $\boldsymbol{b} = b_{\mu\nu}\, \boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu$: The field strength $\boldsymbol{H}$ is invariant under gauge transformations $\boldsymbol{b} \to \boldsymbol{b} + \boldsymbol{d}\wedge\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda} = \Lambda_\alpha \boldsymbol{d}x^\alpha$ is an arbitrary 1-form. In components: The replacement $b_{\mu\nu} \to b_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$ leaves $H_{\mu\nu\alpha}$ unchanged. We have

$${}^*\boldsymbol{J} = m\rho\,{}^*\boldsymbol{v} = \hbar\rho\,{}^*\boldsymbol{d}\theta = m\rho_0\, \boldsymbol{d}\wedge\boldsymbol{b} = m\rho_0\boldsymbol{H}\,, \tag{A.2}$$

$$H_{\alpha\beta\gamma} = \partial_\alpha b_{\beta\gamma} + \partial_\gamma b_{\alpha\beta} + \partial_\beta b_{\gamma\alpha} = (\rho/\rho_0)v^\mu\epsilon_{\mu\alpha\beta\gamma}\,, \tag{A.3}$$

which defines the relation of ρ, θ and $\boldsymbol{b}$. The dual transformation in the conventional sense [13] is obtained if $\rho \to \rho_0$, *i.e.* if we consider compressibility to be negligible in the domain of interest. Then, $\boldsymbol{b}$ has only one degree of freedom corresponding to the order parameter phase θ. For our purposes, this reduction to a single degree of freedom amounts to not considering the cores

of vortices. This is reasonable, because in the dense superfluid helium II the semiclassical U(1) description is not valid in the vortex core.

The superfluid we are dealing with is only Galilei invariant, not Lorentz invariant, so that by writing (A.3) with $\rho = \rho_0$ for spatial and temporal indices separately and taking the Galilean limit, we get the relations

$$-\sqrt{g}\, n_{ijk} = \partial_k b_{ij} + \partial_j b_{ki} + \partial_i b_{jk} = H_{ijk}\,, \tag{A.4}$$

$$\sqrt{g}\, n_{ijk}\, v^i = \partial_k \psi_j - \partial_j \psi_k\,, \tag{A.5}$$

where $\psi_i \equiv b_{0i}$ is the vectorial version of the stream function of classical hydrodynamics [18]. The determinant of the spatial co-ordinate system we are using is designated g and $n_{ijk} = n_{[ijk]} = \pm 1$ is the unit antisymmetric symbol.

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